Full-Stack Quantum Computing

Each problem in this homework has a hand-written portion and a corresponding jupyter notebook for problem 2. The jupyter notebook can be downloaded here. You can run the notebook online at quantum-computing.ibm.com since Datahub doesn't have the qiskit package so these notebooks won't work there. If you need help on how to download your notebook to this platform, refer to the instructions from the other organization I teach for.

1 Measurement

As we learned in lecture, a qubit can be modeled by a two-state system

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

And has to follow the condition that

$$\langle\psi|\psi\rangle = \begin{bmatrix}\bar{\alpha} & \bar{\beta}\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \bar{\alpha}\cdot\alpha + \bar{\beta}\cdot\beta = |\alpha|^2 + |\beta|^2 = 1$$

We can be more specific about this general state $|\psi\rangle$ by modelling it as a function of two variables θ and ϕ. The angles θ and ϕ correspond to rotations on the Bloch Sphere.

$$|\psi\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle = \begin{bmatrix}\cos\theta \\ e^{i\phi}\sin\theta\end{bmatrix} \tag{1}$$

If you're wondering how you'd get i on $|0\rangle$, good question! You can refer here to why we don't model $|\psi\rangle$ in this way

Now, we are going to show that this model holds to the condition we defined above!

(a) Prove that the inner product of $|\psi\rangle$ in (1) is equal to 1. AKA Show that:

$$\langle\psi|\psi\rangle = 1$$

(b) The reason that we stipulate $\langle\psi|\psi\rangle = 1$ is because that means the sum of measuring all the states that our qubit can be in equals 1. So for (1) the probability of measuring $|0\rangle$ is $\cos^2\theta$ and probability of measuring 1 is $\sin^2\theta$. This property leads us to a very easy way to understand the probability of measuring a state given a ket. Remember the values $\cos\theta$ and $e^{i\phi}\sin\theta$ are *amplitudes*, and the amplitudes conjugate squared (i.e. $\bar{\alpha}\cdot\alpha = |\alpha|^2$) is the probability of measuring the corresponding state. So for a qubit state like

$$|\Phi\rangle = \begin{bmatrix}a \\ b \\ c \\ d\end{bmatrix}$$

The probability of measuring $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ is,

$$P(00) = |a|^2 \quad P(01) = |b|^2 \quad P(10) = |c|^2 \quad P(11) = |d|^2$$

Now, what must the probabilities given add up to in order for them to make sense? In other words, what must $\langle\Phi|\Phi\rangle$ add up to? Why? *Don't overthink this lol*

2 Tensor Product Practice

Represent the following tensor products in their vector form. i.e.

$$|-i\rangle = |-\rangle \otimes |i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}$$

Notice: when "−" is in a ket, it is the minus ket, not the minus sign! $-|i\rangle \neq |-i\rangle$ Then, evaluate the probability for measuring each of the states. i.e.

$$\frac{1}{2}\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1\\-i\\-1\\i\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

So there is a 1/4 probability of measuring any of the four states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. Do this analysis on the following states:

(a) $|01\rangle = |0\rangle \otimes |1\rangle$

(b) $|10\rangle = |1\rangle \otimes |0\rangle$

(c) $|{+}0\rangle = |+\rangle \otimes |0\rangle$

(d) $|{-}{+}\rangle = |-\rangle \otimes |+\rangle$

(e) $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$

(f) $\frac{1}{\sqrt{2}}(|{+}11\rangle + |{+}10\rangle) = |+\rangle \otimes \frac{1}{\sqrt{2}}(|11\rangle + |10\rangle)$

Now, make circuits that evaluate to these states and verify your results in the notebook!

Note: When measuring Probabilities in Qiskit using the histogram you won't see a perfect probability spectrum since measurement works by taking many measurements and counting their results. As long as your results are similar to the ones calculate, you should be good!