Topological Error Correction

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Recap From Gottesman Presentation

- Quantum states are prone to decohere, detrimental to running computations on quantum circuits ⟶ Solution : Quantum Error Correction
- Borrow repetition code from classical error correction and adapt it for quantum error correction to correct single-qubit phase-flip and bit-flip errors

$$\text{Classical: } 0 \rightarrow 000, 1 \rightarrow 111$$

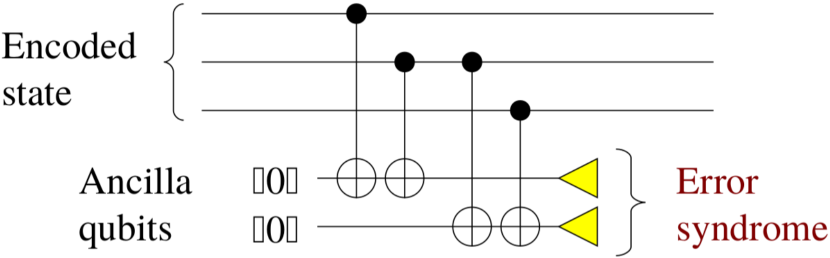
$$\text{Quantum (3-qubit): } \underbrace{|0\rangle \rightarrow |000\rangle, |1\rangle \rightarrow |111\rangle}_{\text{Bit-Flip Errors}}$$

$$\underbrace{|+\rangle \rightarrow |+++\rangle, |-\rangle \rightarrow ---}_{\text{Phase-Flip Errors}}$$

Recap From Gottesman Presentation

Cont.

Figure: 3-Qubit Code Error Correcting Circuit



Recap From Gottesman Presentation

Cont.

- Combining 3-qubit bit-flip and phase-flip error correcting codes gives us 9-qubit code, better known as the Shor Code

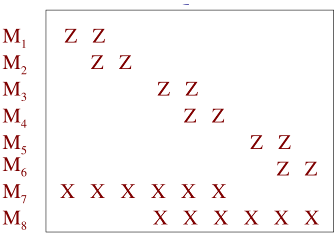
$$\alpha\left|0\right\rangle + \beta\left|1\right\rangle \rightarrow \alpha(\left|000\right\rangle + \left|111\right\rangle)^{\otimes 3} + \beta(\left|000\right\rangle - \left|111\right\rangle)^{\otimes 3}$$

  - Capable of correcting X, Z, and Y errors.
  - By linearity, 9-qubit code is capable of correcting all single-qubit codes.

Recap From Gottesman Presentation

Cont.

- P_n - Pauli Group on n qubits *Ex:* Weight-2 operator $\mathbb{Z} \otimes \mathbb{Z} \otimes \underbrace{I \otimes ... \otimes I}_{\text{7 times}} \in P_9$
- Pauli operators on right determine error syndrome for 9-qubit code → Group of operators known as Stabilizer
- The Stabilizer is an abelian group such that for operator M in the stabilizer, $M \left|\psi\right\rangle = \psi$ for an arbitrary quantum state ψ



M_1	Z	Z							
M_2		Z	Z						
M_3				Z	Z				
M_4					Z	Z			
M_5							Z	Z	
M_6								Z	Z
M_7	X	X	X	X	X	X			
M_8				X	X	X	X	X	X

Figure: Stabilizer for 9-Qubit Code

Topological Quantum Error Correction

Motivation and Definitions

- Allure of Topological Quantum Error Correction: Provides effective way to detect and correct errors that arise on a **local** set of qubits.

Terminology for Quantum Error Correcting Codes of Length n

- Subspace $\mathcal{C}$ of $\mathcal{H}_2^{\otimes n}$
- $\mathcal{E}$ — *Correctable Errors*
- $\mathcal{E}(n, k)$ — Set of operators acting on at most k qubits
- $d(\mathcal{C})$ — Code distance
- $[[n, k, d]]$ — Codes of length n, dimension 2^k, distance d.
- $\frac{k}{n}$ — Encoding Rate

Topological Error Correcting Code

Construction of Codespace

- Family of operators acting on n qubits:

$$\sigma_v := \sigma_{xz} := \bigotimes_{j=1}^{n} i^{x_j z_j} X^{x_j} Z^{z_j}$$

$$x, z \in \mathbb{Z}_2^n, v = (xz) := (x_1, ..., x_n, z_1, ..., z_n)$$

- Commuting operators:

$$\sigma_u \sigma_v = \varphi(u^t \Omega v) \sigma_v \sigma_u$$

Ω is a $2n \times 2n$ matrix over $\mathbb{Z}_2$

$$\varphi(k) := e^{\pi i k}, k \in \mathbb{Z}_2$$

- Set of σ-operators defines Pauli group $P_D(n)$

Topological Error Correcting Code

Construction of Codespace (Cont.)

- Using $P_D(n)$ we can construct stabilizer codes.
- For any subspace $V \subset \mathbb{Z}_2^{2n}$, define $\widehat{V} := \{u \in \mathbb{Z}_2^{2n} | \forall v \in V, v^t \Omega u = 0\}$.
- Define $V_C \subset \mathbb{Z}_2^{2n}$ to be an *isotropic subspace* $(V_C \subset \widehat{V}_C)$
- Stabilizer Group : $\mathcal{S} = \{\sigma_v | v \in B\}$ where B is a basis for V_C.
- Codespace : $\mathcal{C} := \{|\xi\rangle \in \mathcal{H}_2^{\otimes n} | \forall \sigma \in S, \sigma |\xi\rangle = |\xi\rangle\}$
- Code distance :

$$d(\mathcal{C}) = \min_{u \in \hat{V}_C - V_C} |u|$$

Topological Error Correcting Code

Topological Surfaces

- A *surface* is defined as a **compact 2-dimensional manifold.**
- *Examples of topological surfaces:*

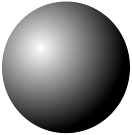


Figure: Sphere

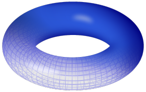


Figure: Torus

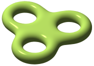


Figure: Triple Torus

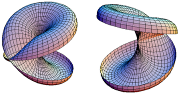


Figure: Projective Plane

Toric Code

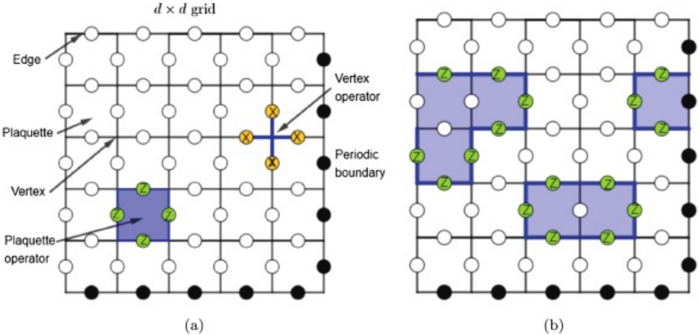


Figure: Toric Code

Topological Error Correcting Code

Graphs and Dual Graphs

- A *graph* Γ is a collection of *vertices* V and *edges* E.
- Consider graph Γ embedded onto surface M.
  - If graph partitions surface into *discs* → Cell Embedding
- *Dual Graph*:
  - Given cell embedding Γ_M, we have a dual embedding Γ^*_M, where inside each face f, we choose a point f^* that serves as a vertex for our new graph Γ^*. We define a new edge e^* as connecting some f^*_1 and f^*_2. Then each vertex v corresponds to a face v^* in Γ^*

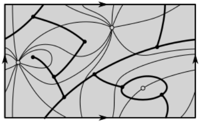
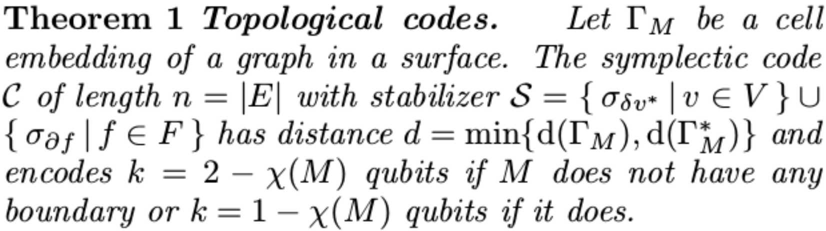


Figure: Dual Graph Example

Topological Error Correcting Code

Formal Definition



Theorem 1 ***Topological codes.*** *Let Γ_M be a cell embedding of a graph in a surface. The symplectic code $\mathcal{C}$ of length $n = |E|$ with stabilizer $\mathcal{S} = \{\, \sigma_{\delta v^*} \mid v \in V \,\} \cup \{\, \sigma_{\partial f} \mid f \in F \,\}$ has distance $d = \min\{\mathrm{d}(\Gamma_M), \mathrm{d}(\Gamma^*_M)\}$ and encodes $k = 2 - \chi(M)$ qubits if M does not have any boundary or $k = 1 - \chi(M)$ qubits if it does.*

Optimal Encoding Rates

- **Interesting Result in Topology:** Every surface can be obtained by taking a combination of spheres, tori, and projective planes.
  - Using this idea, we can define *families* of topological error correcting codes to find optimal encoding rate $\frac{k}{n}$

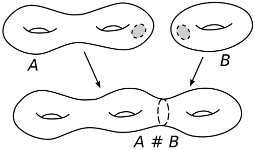


Figure: Connected Sum of Two-Torus and Torus

- *Ex:* Original family of toric codes based on the above result — $[[2d^2, 2, d]]$
- Paper's family of toric codes — $[[d^2 + 1, 2, d]]$

Comparing Surface Codes to Concatenated Codes

- Example of surface code → toric code
- Example of concatenated code → Shor Nine-Qubit Code

Determined that surface codes, but more generally topological error correction is better suited to less reliable quantum computers or other quantum-based devices whereas concatenated codes are better suited to more robust, and less error-prone quantum devices.