

Quantum Singular Value Transformation

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What is Quantum Singular Value Transformation (QSVT)?

Motivation

- Focus is on developing Quantum Algorithms that experience appreciable speed-ups compared to classical counter-parts by exploiting exponential size of unitaries.

Examples

- Quantum Simulation
- Shor's Factoring Algorithm
- Grover's Search Algorithm
- Quantum Random Walks
- HHL Algorithm for Linear Systems of Equations

- Goal of QSVT is to put several quantum algorithmic constructions under the umbrella of a single framework.
- Manages to accomplish this by unifying **Qubitization** and **Quantum Signal Processing**.
 - ▶ This is accomplished with the use of a **Projected Unitary Operator**

Singular Value Decomposition (SVD)

- We know that $\forall A \in \mathbb{C}^{m \times n}$ there exists a pair of unitary matrices ($UU^\dagger = U^\dagger U = I$), $W \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ with values on the diagonal that are non-negative and non-increasing such that $A = W\Sigma V^\dagger$.
- This action is known as **Singular Value Decomposition**.
- Values in diagonal matrix are called **singular values**.
- Columns of V : **right singular vectors**
- Columns of W : **left singular vectors**
- Unitaries V and W represent rotations and Σ represents "stretching".
- Encompasses eigendecomposition ($A = Q\Lambda Q^{-1}$).

Qubitization and Quantum Signal Processing

How QSVT Relates

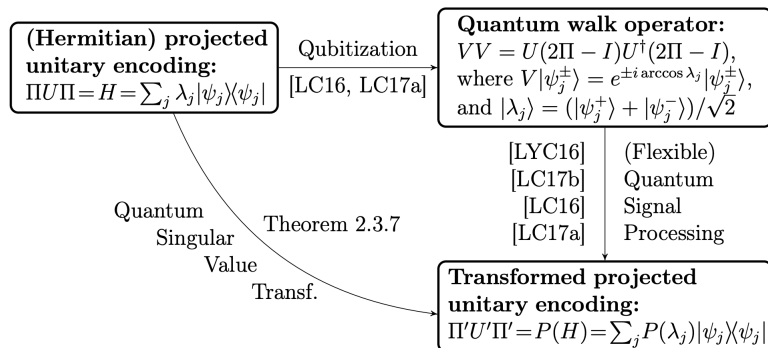


Figure 2.1. Schematic comparison of QSVT with previous approaches.

- Credit: Gilyén et al. (arXiv:1806.01838 [quant-ph])

Qubitization and Quantum Signal Processing

Brief Description

Projected Unitary Encoding (PUE)

- Suppose $\tilde{\Pi}$ and Π are orthogonal projectors and U is a unitary.
- We form the following projected unitary encoding of an arbitrary matrix A :

$$A := \tilde{\Pi}U\Pi$$

- This encoding is called "symmetric" if $\tilde{\Pi} = \Pi$, hence Symmetric Projected Unitary Encoding (SPUE).
- **Qubitization:** Roughly-speaking turns the SPUE of a Hermitian operator H , $H = \Pi U \Pi$, into unitary V , where eigenvector $|\psi\rangle$ of H with eigenvalue $\lambda \mapsto |\psi^\pm\rangle$ with eigenvalues $e^{\pm i \arccos \lambda}$.
- Using **Quantum Signal Processing**, transform spectrum (set of eigenvalues) of V into new circuit U' , which is PUE of transformed operator $P(H)$.

Advantages of QSVT

- Gives us ability to directly apply a transformation directly to the singular values of the Hermitian operator H by improving on formulation of quantum signal processing and borrowing from qubitization.
- Unlike prior methods, QSVT **does not require**:
 - ▶ U to be unitary.
 - ▶ Symmetric encoding of Hermitian H .
 - ▶ Controlled gates to deal with an odd/even polynomial.

Block-Encoded Matrices

- Provides way to efficiently represent large matrices in a quantum computer. Thus, we want a unitary.

$$U = \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix}$$

- ▶ Must have complex matrix A with operator norm $\|A\| \leq 1$
- Workaround

$$U \approx \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix} \implies A \approx \alpha(\langle 0| \otimes I)U(|0\rangle \otimes I)$$

- General aim is to take an arbitrary operator and send it to the upper-left block of some unitary U . Efficiently constructing such unitaries becomes easier if operator is already unitary, matrix is sparse with efficient access to elements, QRAM, etc...

Quantum Singular Value Transformation

Main Result

- Recall qubitization and quantum signal processing approach...
- QSVT expedites this process with far simpler approach: *Let $P : [-1, 1] \rightarrow [-1, 1]$ be a degree- d odd polynomial map (slight adjustment needed for even polynomial maps). Suppose we have the following block-encoding of our qubit operator A :*

$$U = \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \sum_i \zeta_i |w_i\rangle\langle v_i| & \cdot \\ \cdot & \cdot \end{bmatrix}$$

By applying SVD to A , where ζ_i 's are singular values and w_i 's and v_i 's are left and right singular vectors, respectively.

$$\implies U_{\Phi} = \begin{bmatrix} \sum_i P(\zeta_i) |w_i\rangle\langle v_i| & \cdot \\ \cdot & \cdot \end{bmatrix}$$

- With circuit U_{Φ} , we are able to apply polynomial transformation P directly to singular values of qubit operator A . Equivalent to previous method of applying polynomial transformation to eigenvalues of Hermitian H .

Quantum Singular Value Transformation

The Circuit - U_Φ

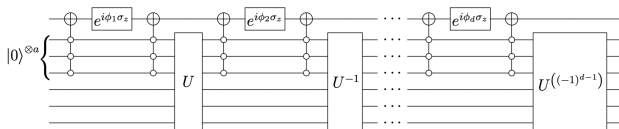
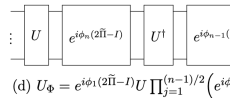
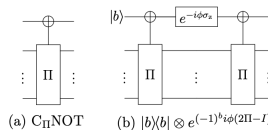


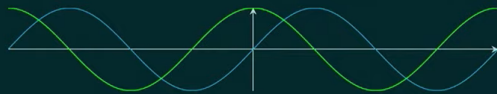
Figure 3.1. Circuits used for quantum singular value transformation. The empty dots denote control by the $|0\rangle$ state, so that the corresponding gate is an “inverted” Toffoli gate, where each qubit is conjugated by an X gate compared to the usual Toffoli gate. The other gates are single-qubit rotations or applications of U or U^\dagger .

● Credit: Gilyén



QSVT's Applications

$\sin(tx), \cos(tx)$:



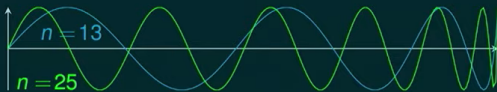
Hamiltonian simulation

$\exp(-\beta x)$:



Gibbs sampling

$T_n(x)$:



Grover search
Ampl. ampl.
Quantum walks

● Credit: Gilyén

QSVT's Applications

The Crux

- Now, we can apply this framework to different applications such as Quantum Simulation, Grover's Search Algorithm, or Quantum Walks.
- Simply need to approximate polynomials related to these applications within the specified bounds and carefully choose Pauli-z rotations based on polynomials.
 - ▶ Polynomials are implemented with degree $\mathcal{O}(\frac{1}{\sqrt{p}} \log(\frac{1}{\epsilon}))$.