Quantum Singular Value Transformation

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What is Quantum Singular Value Transformation (QSVT)?

Motivation

- Focus is on developing Quantum Algorithms that experience appreciable speed-ups compared to classical counter-parts by exploiting exponential size of unitaries.

Examples

- Quantum Simulation
- Shor's Factoring Algorithm
- Grover's Search Algorithm
- Quantum Random Walks
- HHL Algorithm for Linear Systems of Equations

- Goal of QSVT is to put several quantum algorithmic constructions under the umbrella of a single framework.

What is Quantum Singular Value Transformation (QSVT)?

Motivation

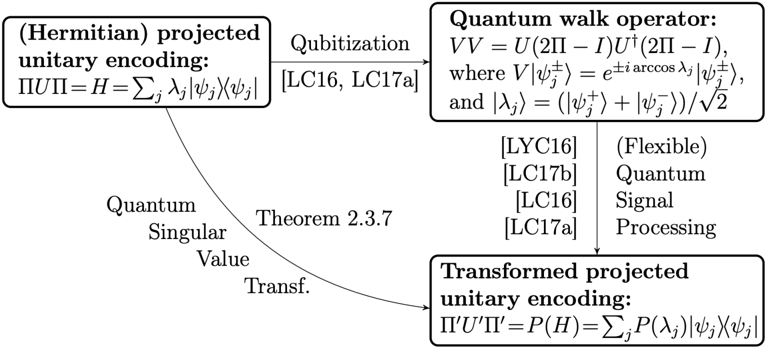
- Theme of spectral/singular value transformation seems to exist across various quantum algorithms:
    - Quantum Simulation
    - Quantum Walks
    - Quantum Linear Systems Algorithms
- Generalize algorithms into a single overarching algorithm using this shared method. This is the key.

Singular Value Decomposition (SVD)

- We know that $\forall A \in \mathbb{C}^{m\times n}$ there exists a pair of unitary matrices $(UU^\dagger = U^\dagger U = I)$, $W \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with values on the diagonal that are non-negative and non-increasing such that $A = W\Sigma V^\dagger$.
- This action is known as Singular Value Decomposition.
- Values in diagonal matrix are called singular values.
- Columns of V: right singular vectors
- Columns of W: left singular vectors
- Unitaries V and W represent rotations and Σ represents "stretching".
- Encompasses eigendecomposition $(A = Q\Lambda Q^{-1})$.

Qubitization and Quantum Signal Processing

How QSVT Relates



(Hermitian) projected unitary encoding:
$\Pi U \Pi = H = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$

Qubitization
[LC16, LC17a]

Quantum walk operator:
$VV = U(2\Pi - I)U^\dagger(2\Pi - I)$,
where $V|\psi_j^\pm\rangle = e^{\pm i \arccos \lambda_j}|\psi_j^\pm\rangle$,
and $|\lambda_j\rangle = (|\psi_j^+\rangle + |\psi_j^-\rangle)/\sqrt{2}$

[LYC16]
[LC17b]
[LC16]
[LC17a]

(Flexible)
Quantum
Signal
Processing

Quantum
Singular
Value
Transf.

Theorem 2.3.7

Transformed projected unitary encoding:
$\Pi' U' \Pi' = P(H) = \sum_j P(\lambda_j) |\psi_j\rangle\langle\psi_j|$

Figure 2.1. Schematic comparison of QSVT with previous approaches.

- Credit: Gilyén et al. (arXiv:1806.01838 [quant-ph])

Qubitization and Quantum Signal Processing

Brief Description

Projected Unitary Encoding (PUE)

- Suppose $\tilde{\Pi}$ and Π are orthogonal projectors and U is a unitary.
- We form the following projected unitary encoding of an arbitrary matrix A:

$$A := \tilde{\Pi} U \Pi$$

- This encoding is called "symmetric" if $\tilde{\Pi} = \Pi$, hence Symmetric Projected Unitary Encoding (SPUE).

- Qubitization: Roughly-speaking turns the SPUE of a Hermitian operator H, $H = \Pi U \Pi$, into unitary V, where eigenvector $|\psi\rangle$ of H with eigenvalue $\lambda \mapsto |\psi^{\pm}\rangle$ with eigenvalues $e^{\pm i arccos \lambda}$.
- Using Quantum Signal Processing, transform spectrum (set of eigenvalues) of V into new circuit U', which is PUE of transformed operator $P(H)$.

Advantages of QSVT

- Gives us ability to directly apply a transformation directly to the singular values of the Hermitian operator H by improving on formulation of quantum signal processing and borrowing from qubitization.
- Unlike prior methods, QSVT does not require:
  - U to be unitary.
  - Symmetric encoding of Hermitian H.
  - Controlled gates to deal with an odd/even polynomial.

Block-Encoded Matrices

- Provides way to efficiently represent large matrices in a quantum computer. Thus, we want a unitary.

$$U = \begin{bmatrix} A & . \\ . & . \end{bmatrix}$$

  - Must have complex matrix A with operator norm $||A|| \leq 1$
- Workaround

$$U \approx \begin{bmatrix} A/\alpha & . \\ . & . \end{bmatrix} \implies A \approx \alpha(\langle 0| \otimes I)U(|0\rangle \otimes I)$$

- General aim is to take an arbitrary operator and send it to the upper-left block of some unitary U. Efficiently constructing such unitaries becomes easier if operator is already unitary, matrix is sparse with efficient access to elements, QRAM, etc...

Quantum Singular Value Transformation

Main Result

- Recall qubitization and quantum signal processing approach...
- QSVT expedites this process with far simpler approach: *Let $P : [-1,1] \to [-1,1]$ be a degree-d odd polynomial map (slight adjustment needed for even polynomial maps). Suppose we have the following block-encoding of our qubit operator A:*

  $$U = \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \sum_i \zeta_i |w_i\rangle\langle v_i| & \cdot \\ \cdot & \cdot \end{bmatrix}$$

  By applying SVD to A, where ζ_i's are singular values and w_i's and v_i's are left and right singular vectors, respectively.

  $$\implies U_\Phi = \begin{bmatrix} \sum_i P(\zeta_i) |w_i\rangle\langle v_i| & \cdot \\ \cdot & \cdot \end{bmatrix}$$

- With circuit U_Φ, we are able to apply polynomial transformation P directly to singular values of qubit operator A. Equivalent to previous method of applying polynomial transformation to eigenvalues of Hermitian H.

Quantum Singular Value Transformation

The Circuit - U_Φ

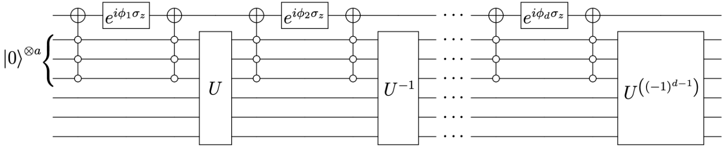
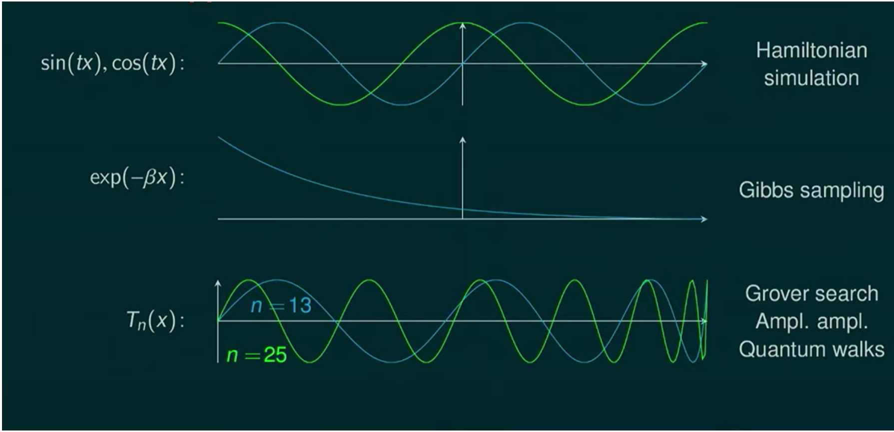


Figure 3.1. Circuits used for quantum singular value transformation. The empty dots denote control by the $|0\rangle$ state, so that the corresponding gate is an "inverted" Toffoli gate, where each qubit is conjugated by an X gate compared to the usual Toffoli gate. The other gates are single-qubit rotations or applications of U or $U^\dagger$.

- Credit: Gilyén

QSVT's Applications



$\sin(tx), \cos(tx)$: Hamiltonian simulation

$\exp(-\beta x)$: Gibbs sampling

$T_n(x)$: Grover search
Ampl. ampl.
Quantum walks

- Credit: Gilyén

QSVT's Applications

The Crux

- Now, we can apply this framework to different applications such as Quantum Simulation, Grover's Search Algorithm, or Quantum Walks.
- Simply need to approximate polynomials related to these applications within the specified bounds and carefully choose Pauli-z rotations based on polynomials.
  - Polynomials are implemented with degree $\mathcal{O}(\frac{1}{\sqrt{p}}log(\frac{1}{\epsilon}))$.