Overview of Quantum Algorithms

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Spring 2022

Why should we care about Quantum Algorithms?

From paper...

Quantum computers are designed to outperform standard computers by running quantum algorithms. - *Ashley Montanaro npj Quantum Information (2016)*

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  - Cryptography (i.e. Shor's Factoring Algorithm)
  - Search and Optimization
  - Simulation of Quantum Systems (i.e. Feynman's motivation)
  - Solving linear systems of equations

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  - Simulation of Quantum Systems (i.e. Feynman's motivation)
  - Solving linear systems of equations
  - ...and more!

Topics

- Measuring Quantum Speed-up
- Grover's Search and the Hidden Subgroup Problem
- Search and Optimization
- Quantum Simulation
- Quantum Walks
- Solving Linear Systems of Equations
- NISQ-Era Algorithm Development
- Zero-Qubit Applications?

How do we know Quantum Algorithms are faster?

Defining and Measuring Quantum Speedup

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Computational Complexity Classes for Quantum Algorithms

- P - Deterministic classical computer, poly-time
- BPP - Probabilistic classical computer, poly-time
- BQP - Quantum computer, poly-time
- NP - Deterministic classical computer, check in poly-time
- QMA - Quantum computer, check in poly-time

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- Why do we care: RSA!

Example (Classical vs. Quantum)

- Classical - 768-bit number, supercomputers over 2 years $\longrightarrow$ 10^20 operations.
- Quantum - 2000-bit number, quantum computer with billion qubits (I know, scary), running for a day

The Famed Shor's Algorithm

The Hidden Subgroup Problem

- Shor's approach based on special case of solution to Hidden Subgroup Problem

Definition (The Hidden Subgroup Problem)

Suppose we're given a group G and a function $f : G$ for some finite set S. Suppose f has the following property: there exists a subgroup $H \subseteq G$ such that f is constant within each coset, and distinct on different cosets. Find this H.

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- Shor's algorithm solves the above problem for the case that $G = \mathbb{Z}$.

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Given we are able to evaluate an arbitrary function of the form $f : \{0,1\}^n \to \{0,1\}$ find x such that $f(x) = 1$, if such an x exists.

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- Quantum - $O(\sqrt{N})$ evalutions of f
  - Slight drawback - Fails with probability ϵ (arbitrary small)

Applications of Grover's Search Algorithm

The Complexity Class NP

- Recall from above...

NP Complexity Class

Problems whose solutions can be checked in polynomial time on a deterministic, classical computer

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  - If answer "yes" $\rightarrow$ have certificate of proof
  - If "no" $\rightarrow$ no certificate

Performance of Grover's Search Algorithm

Circuit SAT

Example (NP-complete Circuit Satisfiability Problem)

- Setup: Electronic circuit comprised of AND, OR, and NOT gates. Take in n bits, output 1 bit.
- Goal: Determine if there exists an input such that the output is 1.

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  - Not exponential, but quadratic speed up achievable.

Generalizing Grover's Algorithm - Amplitude Amplification

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- Classical approach - Run $\mathcal{A}$ many times over and check each output using $f \rightarrow O(1/\epsilon)$ evaluations of f
- Amplitude Amplification - $O(1/\sqrt{\epsilon})$ evaluations of f (Quadratic speed up!)

Applications of Grover's Algorithm and Amplitude Amplification

Example (Applications)

- Finding the minimum of unsorted list of N integers.
- Determining if graph of N vertices is connected.
- Pattern matching
  - Music, speech, bioinformatics, etc...

Adiabatic Optimization

- Alternate approach to quantum combinatorial optimization $\rightarrow$ can solve any constraint satisfaction problem (CSP)

Formulation of Solution

- Prepare quantum state in equal superposition of all solutions to CSP
- Evolve Hamiltonian as slowly as possible with respect to energy gap $\rightarrow$ keeps quantum state in ground state
- Arrive at ground state maximizing most number of constraints $\rightarrow$ solution

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- No well-defined or rigorous worst-case upper bound on run-time.
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- No well-defined or rigorous worst-case upper bound on run-time.
- Effect: Can define problems that take quantum adiabatic algorithm exponential time
- Adiabatic algorithm can be applied on universal quantum computer
  - D-Wave Systems' Quantum Annealers demonstrated outperformance of classical solvers in certain instances.
  - Such systems may not remain in ground-state

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Quantum Simulation

Inspiration from Feynman (Also our favorite quote)

Nature isn't classical, dammit, and if you want to make a simulation of nature, **you'd better make it quantum mechanical**, *and by golly it's a wonderful problem, because it's so easy*

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- Concretely for Quantum Computing: Given a Hamiltonian H describing a physical system and an initial state $|\psi\rangle$, measure properties of interest of time-evolved state $|\psi_t\rangle = e^{iHt} |\psi\rangle$
- Applications include quantum chemistry, superconductivity, metamaterials, and high-energy physics

The Two Approaches to Quantum Simulation

- Digital Quantum Simulation
  - Assumes possession of practical, fault-tolerant, error-correcting quantum computer and use it to run quantum simulation
- Analog Quantum Simulation
  - Approximate behavior of one physical system by constructing another.

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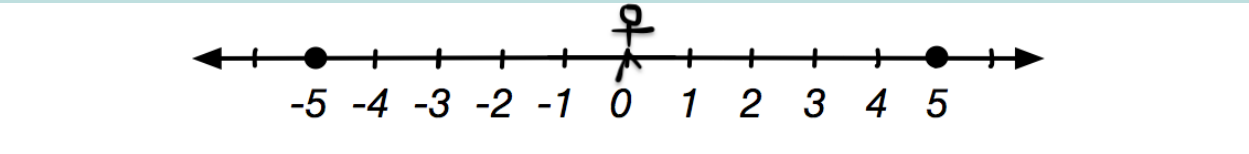
Example (Coined Random Walk)

Imagine you're standing at 0 of the real line and you flip a coin.



- Heads - Go to the right 1
- Tails - Go the left 1

Quantum Walks

"Quantumness" applied to Random Walks

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Graphs for which Quantum Walks have Advantage

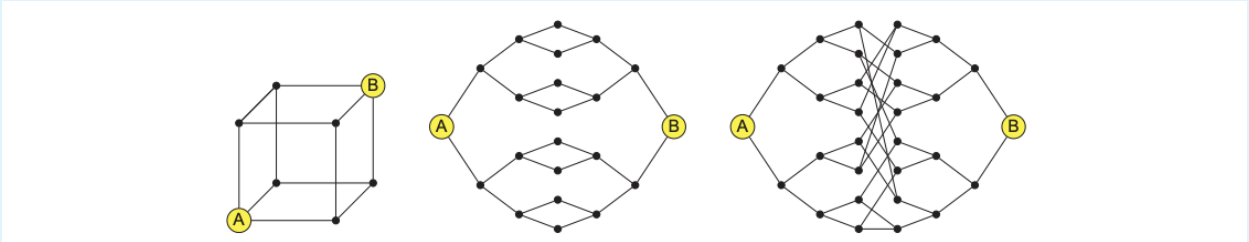

Figure: Quantum walks exhibit faster hitting times when traversing hybercubes, 'glued trees' graphs, and 'glued trees' graphs with a random cycle inserted

Application of Quantum Walks

Example (Evaluation of Boolean Formulae)

Boolean Formulae *on N binary inputs* $x_1, ..., x_N$ *is a tree whose internal vertices represent logical operations that are applied to their children.* N *leaves are labeled with these* N *binary inputs*

- Classical Algorithm - $N^{0.753...}$ worst case
- Quantum Algorithm - $O(N^{1/2})$
- Special case where quantum speed up is realized $\rightarrow$ evaluation of AND-OR trees.

Application of Quantum Walks

Cont.

Example (Algorithms Based on Markov Chains)

Discrete-Time Markov Chain *is a stochastic process whose evolution is determined in terms of a transition matrix*

- Quantum Algorithms have general speed-up over classical algorithms employing Markov Chains

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- Output is n-dimensional quantum state whose properties of interest we can measure.

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